Young Diagrams and Miscellaneous Results

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Abstract

We'll begin by building upon the previous lecture to prove the equivalence between $Spec(n)$ and $Cont(n)$. From there, we'll move on to provide two descriptions of V^λ by constructing a basis and showing how the Coxeter generators act on said basis. Lastly, we'll prove some miscellaneous results such as: a bound on the multiplicity of irreducible representations and a description of the centralizer $Z(l,k)$.

1 Review

Definition. The *Coexeter generators* of S_n are the elements of the form:

$$s_i = (i,\ i+1)$$

Definition. The *Spectrum* of n is the set

$$\text{Spec}(n) = \{\alpha(v) \mid v \in \mathscr{Y}_n\}$$

where $\alpha(v)$ denotes the weight of v and $\mathscr{Y}_n$ is the young basis for S_n. We furthermore define an equivalence relationship $\sim$ on $\text{Spec}(n)$ such that for $\alpha, \beta \in \text{Spec}(n)$ we say $\alpha \sim \beta$ if v_α, v_β are in the same Gelfand-Tetslin basis.

In Ryan's talk, he proved the following important proposition about how the coexeter generators act on the elements of the Spectrum:

Theorem 1.1. *Suppose*

$$\alpha = (a_1, \dots, a_i, a_{i+1}, \dots, a_n) \in \text{Spec}(n)$$

then if $a_{i+1} \neq a_i \pm 1$ we have:

$$\alpha' := s_i \cdot \alpha = (a_1, \dots, a_{i+1}, a_i, \dots, a_n)$$

and over the vector space with basis $v_{\alpha'}, v_\alpha$ for fixed v_α and some choice of $v_{\alpha'}$

$$s_i = \begin{pmatrix} \frac{1}{a_{i+1}-a_i} & 1 - \frac{1}{(a_{i+1}-a_i)^2} \\ 1 & \frac{1}{a_i - a_{i+1}} \end{pmatrix}$$

We will use the above theorem extensively in section 3 to give a basis description of V^λ. However, we will first begin by extending the results from Myeonhu's talk on content vectors. For a review:

Definition. The n-th *Content Vector*, denoted $\text{Cont}(n)$, is defined to be the set of all tuples $(a_1, \dots, a_n)$ such that

1. $a_1 = 0$
2. If $a_q > 0$ there exists an $i < q$ such that $a_i + 1 = a_q$. If $a_q < 0$ there exists an $i < q$ such that $a_i - 1 = a_q$.
3. If $a_p = a_q = a$ for some $p < q$, then there exists $p < i, j < q$ such that $a_i = a - 1$ and $a_j = a + 1$.

We furthermore define an equivalence relation $\approx$ on $\text{Cont}(n)$ such that if $\alpha, \beta \in \text{Cont}(n)$ we say $\alpha \approx \beta$ if α, β are equal as sets (i.e. contain the same elements).

Lemma 1.2. *If $\alpha, \beta \in \mathrm{Spec}(n)$ have $\alpha \sim \beta$, then $\alpha \approx \beta$.*

Lemma 1.3. *For all positive integers n, we have $\mathrm{Spec}(n) \subset \mathrm{Cont}(n)$*

Lemma 1.4. *If $\alpha \in \mathrm{Spec}(n)$ and $\beta \in \mathrm{Cont}(n)$ satisfy $\alpha \approx \beta$, then $\beta \in \mathrm{Cont}(n)$ and $\alpha \sim \beta$.*

In this talk, we will begin by using these result to prove equivalence between $\mathrm{Spec}(n)/\sim$ and $\mathrm{Cont}(n)/\approx$

2 Equivalence between $\mathrm{Spec}(n)$ and $\mathrm{Cont}(n)$

Theorem 2.1. *For all positive integers n, we have $\mathrm{Spec}(n) = \mathrm{Cont}(n)$. Furthermore, the two equivalence relations $\sim, \approx$ are equivalent.*

Proof. Consider an equivalence class E in $\mathrm{Cont}(n)/\approx$. If a single element of this equivalence class lies in $\mathrm{Spec}(n)$, then by lemma 1.4, every element of E must lie in $\mathrm{Spec}(n)$ and be equivalent under $\sim$. Furthermore, by lemmas 1.2 and 1.3, any equivalence class in $\mathrm{Spec}(n)$ must be an equivalence class in $\mathrm{Cont}(n)$ and furthermore, all of the elements must be equivalent under $\approx$.

In total, the equivalence classes of $\mathrm{Spec}(n)/\sim$ are a subset of those of $\mathrm{Cont}(n)/\approx$. Thus if we can show:

$$|\mathrm{Spec}(n)/\sim| = |\mathrm{Cont}(n)/\approx|$$

the two sets must contain precisely the same equivalence classes and thus $\mathrm{Spec}(n) = \mathrm{Cont}(n)$ and $\sim=\approx$. We will now prove precisely this.

Let $p(n)$ denote the number of partitions of n. Recall that there is a bijection between $\mathrm{Cont}(n)/\approx$ and the set of young diagrams with n boxes. Note that there is a bijection between the set of young diagrams and the partitions of n defined by sending a young diagram D to the partition:

$$D_1 + D_2 + \cdots + D_r$$

where r is the number of rows in D and D_i is the number of boxes in the n-th row. Thus we have:

$$|\mathrm{Cont}(n)/\approx| = p(n)$$

Now by definition:

$$|\mathrm{Spec}(n)/\sim| = \#\{\hat{S}_n\}$$

where the latter denotes the set of irreducible representations of $\{\hat{S}_n\}$. Now, by a well-know result, $\#\{\hat{S}_n\}$ is simply the number of conjugacy classes of n. As conjugacy preserves cycle type and all cycles of the same length are conjugate, each conjugacy class of S_n will only depend on the length of cycles it contains. Thus we can send each conjugacy class C to the partition:

$$C_1 + C_2 + \cdots + C_r$$

where C_i is the length of the i-th cycle of C when ordered in decreasing order. For example,

$$(1345)(27)(6)(9) \mapsto 4 + 2 + 1 + 1$$

This map is a bijection, thus

$$|\mathrm{Spec}(n)/\sim| = \#\{\hat{S}_n\} = p(n)$$

Hence we have shown:

$$|\mathrm{Spec}(n)/\sim| = |\mathrm{Cont}(n)/\approx|$$

as desired. $\square$

Remark. As $\mathrm{Cont}(n)$ is bijective with the space of path to the n-th row in the Young Graph and $\mathrm{Cont}(n)/\approx$ is bijective with n-the row of the Young Graph, it follows from this result that the Young Graph is the branching graph of the symmetric group. Furthermore, the space of paths is bijective to the elements of the young basis.

3 Young Formulas

We'll begin by defining an important quantity to be used from now on:

Definition. Let $s \in S_n$, the *inversion number* of s is defined:

$$l(s) = \#\{(i,j) \mid i < j \text{ and } s(i) > s(j)\}$$

Remark. It is a well-known fact that for $s \in S_n$, $l(s)$ is equivalently the smallest r such that there exists coxeter generators $a_1, \ldots, a_r$ for which $s = a_1 \cdots a_r$

We also want to be able to define the inversion number $l(T)$ for young-tableaux T. Suppose $T \in \text{Tab}(\lambda)$ where $\lambda = (\lambda_1, \ldots, \lambda_r)$. As in proposition 3.6 of Myeonhu's talk, we define T_0 to be the young tableaux with diagram λ and monotone numeration (i.e. the tableaux where we sort the boxes first by row number and then by column number with smaller rows and columns coming first and then label them increasing order with the positive integers $1, 2, 3, 4, \ldots$). Let s be the permutation that sends T_0 to T. Then we define:

$$l(T) = l(s)$$

Following Myeonhu's proof of proposition 3.6, we can see that in the transformation from $T \to T_0$ contains precisely $l(T)$ admissible transpositions were used.

We'll now give a specific choice of basis for the given representation V^λ and prove some properties that is satisfies. Let v_{T_0} be any nonzero vector corresponding to the tableaux $T_0 \in \text{Tab}(\lambda)$. For all other $T \in \text{Tab}(\lambda)$ we define:

$$v_T = P_T \cdot s \cdot v_{T_0} \tag{1}$$

We'll first prove the following lemma:

Lemma 3.1. *Let $T \in \text{Tab}(\lambda)$ and let $s \in S_n$ be such that $T = sT_0$. Then there exists rational numbers γ_R such that*

$$s \cdot v_{T_0} = v_T + \sum_{R \in \text{Tab}(\lambda), l(R) < l(T)} \gamma_R v_R$$

Proof. We'll prove this by induction on $l(T)$. The base case $l(T) = 0$ is clear. Now suppose this holds for all T with $l(T) \le k$. Suppose $U \in \text{Tab}(\lambda)$ has $\lambda(U) = k+1$. There exists a $U' \in \text{Tab}(\lambda)$ and a coxeter generator s_j such that $\lambda(U') = k$ and $U = s_j U'$. Let s be such that $sT_0 = U'$. We have for some γ_R:

$$s \cdot v_{T_0} = v_{U'} + \sum_{R \in \text{Tab}(\lambda), l(R) < l(U')} \gamma_R v_R$$

multiplying both sides by s_j:

$$s_j s \cdot v_{T_0} = s_j \cdot v_{U'} + \sum_{R \in \text{Tab}(\lambda), l(R) < l(U')} \gamma_R s_j v_R$$

Now, by theorem 1.1, for each R there exists γ' and a_{i+1}, a_i such that

$$s_j v_R = \frac{v_R}{a_{i+1} - a_i} + \gamma' v_{s_j R}$$

As $l(s_j R) \le l(R) + 1 < l(U') + 1 = l(U)$, there exists γ'_R such that

$$s_j s \cdot v_{T_0} = s_j \cdot v_{U'} + \sum_{R \in \text{Tab}(\lambda), l(R) < l(U)} \gamma'_R v_R$$

Similarly there exists γ', a'_{i+1}, a'_i such that,

$$s_j v_{U'} = \frac{v_{U'}}{a'_{i+1} - a'_i} + \gamma' v_U$$

The first term can be absorbed into the summation so that there exists $\bar{\gamma}_R$ such that

$$s_j s \cdot v_{T_0} = \gamma' \cdot v_U + \sum_{R \in \text{Tab}(\lambda), l(R) < l(U)} \bar{\gamma}_R v_R$$

Taking P_U of both sides gives:

$$v_U = P_U s_j s \cdot v_{T_0} = \gamma' \cdot v_U$$

Thus $\gamma' = 1$ and our induction is complete. $\square$

Theorem 3.2. *The previously described basis $\{v_T\}$ of V^λ has the property that the Coxeter generators s_i act such that if $T' = s_i T$, $l(T') > l(T)$ and*

$$\alpha(T) = (a_1, \ldots, a_n) \in \mathrm{Cont}(n)$$

then we have:

$$s_i \cdot v_T = v_{T'} + \frac{1}{a_{i+1} - a_i} v_T$$

and

$$s_i \cdot v_{T'} = \left(1 - \frac{1}{(a_{i+1} - a_i)^2}\right) v_T - \frac{1}{a_{i+1} - a_i} v_{T'}$$

Proof. We will use the same notation as in the prelude to this theorem. Let v_{T_0} be a nonzero vector corresponding to the tableaux $T_0 \in \mathrm{Tab}(\lambda)$. For each tableaux $T \in \mathrm{Tab}(\lambda)$, if $T = sT_0$, define v_T to be:

Now, by theorem 1.1 implies that for all T and s_i we have

$$s_i \cdot v_T = c_{T,s_i} v_{T'} + \frac{1}{a_{i+1} - a_i} v_T \tag{2}$$

and

$$s_i \cdot c_{T,s_i} v_{T'} = \left(1 - \frac{1}{(a_{i+1} - a_i)^2}\right) v_T - \frac{c_{T,s_i}}{a_{i+1} - a_i} v_{T'}$$

for some constants c_{T,s_i}. Now suppose $s \in S_n$ is such that $T = sT_0$. By lemma 3.1 for some rationals γ_R, γ'_R,

$$s v_{T_0} = v_T + \sum_{l(R) < l(T)} \gamma_R v_R$$
$$s_i s v_{T_0} = v_{T'} + \sum_{l(R) < l(T')} \gamma'_R v_R$$

multiplying the former by s_i and equating the two gives, as $l(T) < l(T')$, that for some rationals $\bar{\gamma}_R$ we have:

$$s_i v_T = v_{T'} + \sum_{l(R) < l(T')} \bar{\gamma}_R$$

applied to equation 2 this implies $c_{T,s_i} = 1$, which gives us the desired result. □

Remark. We can also scale the vectors differently to get an orthogonal basis. In this case the action of s_i becomes:

$$\begin{pmatrix} r^{-1} & \sqrt{1 - r^{-2}} \\ \sqrt{1 - r^{-2}} & r^{-1} \end{pmatrix}$$

with $r = a_{i+1} - a_i$

4 Miscellaneous Result I

For this following theorem we will index irreducible representations by the corresponding young diagram.

Theorem 4.1. *Let μ be a young diagram with n boxes and λ be a young diagram with $n + k$ boxes. The multiplicity of an irreducible representation π_μ of S_n in a representation π_λ is equal to the number of paths between μ and λ. Thus the multiplicity is 0 if there are no paths, and it is fewer than $k!$ in all cases.*

Proof. The multiplicity being the number of paths follows directly from the equivalence between the branching graph and the young graph. Thus we just need to show the bound of $k!$. Note that the diagram λ will be the diagram μ with k additional boxes. Each path from μ to λ will correspond to a labelling of these boxes with the numbers $n + 1, \ldots, n + k$ by our definition of young tableaux. Furthermore, each path will have a different labelling. As there are $k!$ ways to label the k boxes with these k numbers, there are at most $k!$ paths. Thus the multiplicity is at most $k!$. □

5 Miscellaneous Result II

For this section we will use $\mathbb{I}_k$ to denote the identity in S_k. We first define the projection $\tilde{p}_n : S_n \to S_{n-1}$ as follows: Take $\sigma \in S_n$ and write it as a product of cycles

$$\sigma = c_1 c_2 \cdots c_r$$

then define $\tilde{p}_n$ to simply be the element that results from removing n from the cycle that it lies in. For instance:

$$\tilde{p}_3((13)) = \mathbb{I}_3$$
$$\tilde{p}_5((125)(34)) = (12)(34)$$
$$\tilde{p}_6((124635)) = (12435)$$

We now have the following lemma on $\tilde{p}_n$:

Lemma 5.1. *We have:*

1. $\tilde{p}_n(\mathbb{I}_n) = \mathbb{I}_{n-1}$
2. $\tilde{p}_n|_{S_{n-1}} = \mathrm{id}_{S_{n-1}}$
3. $\tilde{p}_n(g_1 h g_2) = g_1 \tilde{p}_n(h) g_2 \quad$ *for* $\;g_1, g_2 \in S_{n-1}, h \in S_n$

Proof. Properties $1, 2$ are clear. Suppose i, j are such that $h(i) = n$ and $h(n) = j$. Then $\tilde{p}_n(h)$ is identical to h except $\tilde{p}_n(h)(i) = j$ and it is undefined at n. Let r be such that $g_2(r) = i$. Then as $g_2(n) = n$,

$$\tilde{p}_n(hg_2)(r) = h(g_2(n)) = j = \tilde{p}_n(h)g_2(r)$$

At all other values it is clearly the same, thus $\tilde{p}_n(hg_2) = \tilde{p}_n(h)g_2$. Applying similar reasoning to g_1 gives the desired result. □

By linearity, we can extend the map $\tilde{p}_n$ to a map of the group algebras:

$$p_n : \mathbb{C}[S_n] \to \mathbb{C}[S_{n-1}]$$

Recall that $Z(n-1,1)$ is the centralizer of $\mathbb{C}[S_{n-1}]$ in $\mathbb{C}[S_n]$. Using the above notation, we have the following theorem.

Theorem 5.2.
$$p_n^{-1}(\{c\mathbb{I}\}) \cap Z(n-1,1) = \{aX_n + b\mathbb{I}\}$$
where $\{c\mathbb{I}\}$ *denotes the vector space spanned by the identity,* $Z(n-1,1)$ *denotes the centralizer of* S_{n-1} *in* $\mathbb{C}[S_n]$*,* X_n *denoted the* n*-th Young-Jucys-Murphy element, and* $\{aX_n + b\mathbb{I}\}$ *denotes the set spanned by* X_n *and* $\mathbb{I}$*.*

Proof. Recall that
$$Z(n-1,1) = \langle Z_{n-1}, X_n \rangle$$
Thus suppose $g \in Z(n-1,1)$. Then we can write: $g = z + bX_n$ for some $z \in Z_{n-1}$. If

$$g \in p_n^{-1}(\{c\mathbb{I}\}) \cap Z(n-1,1)$$

then we must further have for some k:

$$k\mathbb{I} = p_n(g) = p_n(z) + p_n(bX_n)$$

As p_n fixes elements in S_{n-1} we have $p_n(z) = z$. Furthermore, by its definition, $p_n(X_n) = b(n-1)\mathbb{I}$. Thus we have
$$k\mathbb{I} = z + b(n-1)\mathbb{I}$$
which is satisfied if and only if $z = (k - b(n-1))\mathbb{I}$. Thus the elements of $p_n^{-1}(\{c\mathbb{I}\}) \cap Z(n-1,1)$ are precisely those of the form:
$$aX_n + b\mathbb{I}$$
for constants a, b as desired. □

Remark. This allows us to define a fancy group:

$$\mathfrak{S} = \lim_{\leftarrow}(S_n, \tilde{p}_n)$$

In the same vein as the other theorems, we have:

Theorem 5.3. *The centralizer:*

$$Z(n,k) := \mathbb{C}[S_{n+k}]^{\mathbb{C}[S_n]}$$

is generated by the center $Z(n)$ *of* $\mathbb{C}[S_n]$*, the group* S_n *permuting the elements* $n+1,\dots,n+k$*, and the YJM-elements* $X_{n+1},\dots,X_{n+k}$.

Proof. The proof follows the same structure as the proof of theorem 4.8 in Micah's talk. □